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INTEGRATED TRAPEZOIDAL COLLOCATION METHOD FOR SOLVING THIRD ORDER INTEGRO-DIFFERENTIAL EQUATIONS

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Abstract
In this paper, the Integrated Trapezoidal Collocation Method for solving third order linear and nonlinear Integro-Differential Equations is presented. The highest derivative that appeared in the problems considered is approximated by the Power series and Canonical polynomials which are constructed for the problem considered. Thus, the assumed solution is then integrated successively to obtain the lower order derivatives contained in the problem while the Trapezoidal rule is then applied on the first order derivative to obtain the unknown function itself. These derivatives and the unknown function are then substituted into the given problem and after simplification, the resulting equation is collocated at some equally spaced interior points in the intervals of consideration, this leads to system of linear algebraic equations which are then solved by 'Maple 18' package to obtain the values of the unknown constants that are contained in the assumed solution. These values are then substituted back into the unknown function to obtain the approximate solution required. Numerical experiments show that the method is easy to apply and of high accuracy. From the results presented in Tables 1-3 and Figures 1-3, it is observed that the two basis functions produce similar results with Chebyshev polynomials as basis functions and that the method yields the desired accuracy when compared with the exact solutions.

Keywords: Integrated Collocation Method, Trapezoidal rule, Power series, Canonical polynomials, Integro-Differential Equations.

1. Introduction.

The study of numerical solutions of Integro-Differential Equations and other functional equations have been very pronounced in recent years. Integro-Differential Equations are either of Fredholm, Volterra or Fredholm-Volterra type [1]. They occur in many fields of study, including fluid mechanics, biological processes, chemical kinetics, engineering and economics [2]. Obtaining the solutions of Integro-Differential Equations particularly nonlinear type in closed form is generally difficult, therefore, the inevitability of the solutions in numeric or approximate form [3].

In recent time, many numerical analysts have examined the numerical solutions of third order Integro-Differential Equations. Some of the methods applied include: Taylor Polynomial Solution [4], Chebyshev Polynomial Approach [5], Block-pulse Functions and Operational Matrices [6], Semi-Orthogonal Spline Wavelets Approximation [7] and Lagrange Interpolation Method [8]. Other methods applied to solve Integro-Differential Equations are: Differential Transform Method (DTM) [9], Sine-Cosine Wavelets Method [10] and Pseudo Spectra Method [11].

For the purpose of our discussion, we shall consider the general third order linear and nonlinear Integro-Differential Equation of the following types:
(i) Fredholm Integro-Differential Equation(FIDE)

$$\sum_{i=0}^{3} P_i y^{(i)}(x) + \int_a^b k(x,t)y(t)dt = f(x) \tag{1}$$

(ii) Fredholm-Volterra Integro-Differential Equation(FVIDE)

$$\sum_{i=0}^{3} P_i y^{(i)}(x) + \int_a^b k(x,t)y(t)dt + \int_a^x k(x,t)y(t)dt = f(x) \tag{2}$$

Here, equations (1) and (2) are subjected to the conditions

$$y^{(i)}(a) = \alpha_j, \quad i = 0,\ 1,\ 2, ...,\ (n-1) \tag{3}$$

and

$$y^{(i)}(b) = \beta_j, \quad i = n,\ (n+1),\ (n+2), ...,\ (k-1) \tag{4}$$

where, P_i $(i \geq 0)$ are constants, $k(x,t)$ and $f(x)$ are given smooth (i.e. differentiable and integrable) functions in $[a,\ b]$, $y^{(i)}(x)$ denotes the ith derivative of $y(x)$, $\alpha_j :\ 0 \leq j \leq (n-1)$, $\beta_j :\ n \leq j \leq (k-1)$, are real finite constants and $y(x)$ is the unknown function to be determined.

2. Methodology and Techniques.
In this section, we applied the Integrated Trapezoidal Collocation Method to solve both linear and non-linear third order Integro-Differential Equations.

To illustrate the basic concept of the method, we consider the following general non-linear system:

$$L[y(x)] = N[y(x)] + M[y(x)] + f(x) \tag{5}$$

where, L, M are linear operators, N is a non-linear operator and $f(x)$ is a given smooth function.

For non-linear problem, we employed the Taylor's series linearization scheme to obtain a linear approximation at $t_0 = 0$.

2.1 Taylor's series linearization scheme
Let

$$G_y = y^{(n)}(t) \tag{6}$$

be the non-linear expression contained in the problem considered, expanding the right hand side of equation (6) in Taylor's series around the point t_0, we obtained

$$G_y \equiv y(t) + (t-t_0)y'(t) + \frac{(t-t_0)^2 y''(t)}{2!} + \frac{(t-t_0)^3 y'''(t)}{3!} + ... + \frac{(t-t_0)^n y^{(n)}(\xi)}{n!} \tag{7}$$

Putting $t_0 = 0$ in equation (7), we have

$$G_y \equiv y(t) + ty'(t) + \frac{t^2}{2!}y''(t) + \frac{t^3}{3!}y'''(t) + ... + \frac{t^n}{n!}y^{(n)}(\xi) \tag{8}$$

Truncating equation (8) at the term containing $y'(t)$, we have

$$G_y \approx y(t) + ty'(t) \tag{9}$$

Therefore, equation (9) is a linear approximation to equation (6).

2.2 Construction of Canonical Polynomials for k^{th} Order IDEs

Consider the general k^{th} order Integro-Differential Equation

$$P_0y(x) + P_1y'(x) + P_2y''(x) + ... + P_ky^{(k)}(x) + \int_a^b k(x,t)y(t)dt = f(x) \tag{10}$$

We defined the operator

$$L \equiv P_k\frac{d^k}{dx^k} + P_{k-1}\frac{d^{k-1}}{dx^{k-1}} + ... + P_1\frac{d}{dx} + P_0 \tag{11}$$

Let

$$L\Phi_r(x) = x^r \tag{12}$$

Then

$$L[L\Phi_r(x)] = Lx^r \tag{13}$$

Therefore,

$$Lx^r \equiv P_k\frac{d^kx^r}{dx^k} + P_{k-1}\frac{d^{k-1}}{dx^{k-1}x^r} + ... + P_1\frac{dx^r}{dx} + P_0x^r \tag{14}$$

$$Lx^r = P_kr(r-1)(r-2)...(r-k+1)x^{r-k} + P_{k-1}r(r-1)(r-2)...(r-k+2)x^{r-k+1} + ... + P_1rx^{r-1} + P_0x^r \tag{15}$$

This implies

$$L[L\Phi_r(x)] = P_kr(r-1)(r-2)...(r-k+1)x^{r-k} + P_{k-1}r(r-1)(r-2)...(r-k+2)x^{r-k+1}$$
$$+ ... + P_1rx^{r-1} + P_0x^r$$

Suppose the inverse L^{-1} exists, then

$$L[L\Phi_r(x)] = P_kr(r-1)(r-2)...(r-k+1)L\Phi_{r-k}(x) + P_{k-1}r(r-1)(r-2)...(r-k+2)L\Phi_{r-k+1}(x)$$
$$+ ... + P_1rL\Phi_{r-1}(x) + P_0L\Phi_r(x)$$
$$x^r = P_kr(r-1)(r-2)...(r-k+1)\Phi_{r-k}(x) + P_{k-1}r(r-1)(r-2)...(r-k+2)\Phi_{r-k+1}(x)$$
$$+ ... + P_1r\Phi_{r-1}(x) + P_0\Phi_r(x)$$

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Therefore,

$$\Phi_r(x) = \frac{1}{P_0}[x^r - P_k r(r-1)(r-2)...(r-k+1)\Phi_{r-k}(x) - P_{k-1} r(r-1)(r-2)...(r-k+2)\Phi_{r-k+1}(x)$$

$$- ... - P_1 r\Phi_{r-1}(x)], \quad P_0 \neq 0 \quad , \quad r \geq 0 \tag{16a}$$

Let $k = 3$ in equation (16a), we have

$$\Phi_r(x) = \frac{1}{P_0}[x^r - P_3 r(r-1)(r-2)\Phi_{r-3}(x) - P_2 r(r-1)\Phi_{r-2}(x) - P_1 r\Phi_{r-1}(x)], r \geq 0 \tag{16b}$$

The Canonical polynomials used in this work are obtained recursively using equation (16b).

2.3 Integrated Trapezoidal Collocation Method by Power Series Approach (ITCMPS)

In order to apply this method to solve equation (1) or (2) together with the initial/boundary conditions given in equations (3) and (4), we assumed the power series approximation given by

$$y'''(x) = \sum_{j=o}^{N} a_j x^j \tag{17}$$

and

$$y''(x) = \int \sum_{j=o}^{N} a_j x^j dx + c_1 \tag{18}$$

$$y'(x) = \int \int \sum_{j=o}^{N} a_j x^j dx dx + c_1 x + c_2 \tag{19}$$

Obtaining $y(x)$ from equation (19), we write

$$y(x) = \int y'(x)dx + c_3 \tag{20}$$

where, C_3 is the constant of integration.

We applied the Trapezoidal rule to evaluate the integral part of equation (20), we write

$$y_{N,n}(x) = \frac{h}{2}[f(x_0) + 2(f(x_1) + f(x_2) + ... + f(x_{n-1})) + f(x_n)] + C_3 \tag{21}$$

where $x_0 = 0$, $x_n = x_0 + nh$, such that $(n \geq 2)$ is an integer, $f(x_n) = y'(x_n)$ and $h = \frac{b-a}{n}$.
Thus, equations (17) – (21) are substituted into equation (10) by selecting $n = 3$, we obtained

$$P_0 y_{N,n}(x) + P_1[\int \int \sum_{j=o}^{N} a_j x^j dx dx + c_1 x + c_2] + P_2[\int \sum_{j=o}^{N} a_j x^j dx + c_1] + P_3 \sum_{j=o}^{N} a_j x^j + \int_a^b k(x,t) y_{N,n}(t)dt = f(x) \tag{22}$$

Expanding and simplifying equation (22), we obtained

$$P_0[a_0W_0^{(3)}(x)+a_1W_1^{(3)}(x)+a_2W_2^{(3)}(x)+....+a_NW_N^{(3)}(x)]$$
$$+P_1[a_0W_0^{(2)}(x)+a_1W_1^{(2)}(x)+a_2W_2^{(2)}(x)+....+a_NW_N^{(2)}(x)]$$
$$+P_2[a_0W_0^{(1)}(x)+a_1W_1^{(1)}(x)+a_2W_2^{(1)}(x)+....+a_NW_N^{(1)}(x)]$$
$$+P_3[a_0W_0^{(0)}(x)+a_1W_1^{(0)}(x)+a_2W_2^{(0)}(x)+....+a_NW_N^{(0)}(x)]$$
$$+\mathbf{C}(X)+\mathbf{G}_1(X,t)=f(x) \tag{23}$$

Here,

$$W_N^{(3)}(x)=\int\int\int x^N dxdxdx \tag{24}$$

$$W_0^{(3)}(x)=\int\int\int dxdxdx \tag{25}$$

$$\mathbf{G}_1(X,t)=\int_a^b k(x,t)y_{N,n}(t)dt \tag{26}$$

and $C(X)$ is the sum of all the expressions containing $c_i \ : i=1,2,3$ and $P_i \ : i=0,1,2,3$. After evaluating the terms involving integrals in equation (23) and with further simplification, we then collocated the left-over at the point $x=x_k$, we obtained

$$P_0[a_0W_0^{(3)}(x_k)+a_1W_1^{(3)}(x_k)+a_2W_2^{(3)}(x_k)+....+a_NW_N^{(3)}(x_k)]$$
$$+P_1[a_0W_0^{(2)}(x_k)+a_1W_1^{(2)}(x_k)+a_2W_2^{(2)}(x_k)+....+a_NW_N^{(2)}(x_k)]$$
$$+P_2[a_0W_0^{(1)}(x_k)+a_1W_1^{(1)}(x_k)+a_2W_2^{(1)}(x_k)+....+a_NW_N^{(1)}(x_k)]$$
$$+P_3[a_0W_0^{(0)}(x_k)+a_1W_1^{(0)}(x_k)+a_2W_2^{(0)}(x_k)+....+a_NW_N^{(0)}(x_k)]$$
$$+\mathbf{C}(X_k)+\mathbf{G}_1(X_k,t)=f(x_k) \tag{27}$$

where,

$$x_k=a+\frac{(b-a)k}{N+2}, \quad k \ = \ 1, \ 2, \ 3, ..., N+1 \tag{28}$$

Putting equation (28) into (27), we obtained (N+1) algebraic equations with (N+4) unknown constants. Three extra equations are obtained using the initial/boundary conditions given in equations (3) and (4). Altogether, we have (N+4) algebraic equations with (N+4) unknown constants. This system of (N+4) algebraic linear equations is put in vector form as $A\underline{X}=\underline{b}$ and then solved using Gaussian 'Maple 18' software to obtain the unknown constants a_j $(j \geq 0)$ and $c_i's$. These values are then substituted into our assumed solution to obtain the approximate solution.

2.4 Integrated Trapezoidal Collocation Method by Canonical Polynomials (ITCMCP)

Again, we applied the Integrated Trapezoidal Collocation method using Canonical polynomials as our basis function to solve equation (1) together with the initial/boundary conditions given in equations (3)

and (4).
We assumed the approximation of the form

$$\frac{d^3y}{dx^3} = \sum_{j=o}^{N} a_j \Phi_j(x) \tag{29}$$

where, $\Phi_j(x)$ are Canonical polynomials obtained from equation (16).
Integrating equation (29) successively, we obtained and

$$\frac{d^2y}{dx^2} = \int \sum_{j=o}^{N} a_j \Phi_j(x) dx + c_1 \tag{30}$$

$$y'(x) = \int \int \sum_{j=o}^{N} a_j \Phi_j(x) dx dx + c_1 x + c_2 \tag{31}$$

Using equations (7) and (8), we obtained the approximation $y_{N,n}(x)$ to the unknown function $y(x)$. Thus, substituting equations (29) - (31) together with the approximation $y_{N,n}(x)$ into equation (10) by selecting $n = 3$, we obtained

$$\begin{gathered} P_0 y_{N,n}(x) + P_1 [\int \int \sum_{j=o}^{N} a_j \Phi_j(x) dx dx + c_1 x + c_2] + P_2 [\int \sum_{j=o}^{N} a_j \Phi_j(x) dx + c_1] \\ + P_3 \sum_{j=o}^{N} a_j \Phi_j(x) + \int_a^b k(x,t) y_{N,n}(t) dt = f(x) \end{gathered} \tag{32}$$

Expanding and simplifying equation (32), we obtained

$$\begin{gathered} P_0[a_0\Phi_0^{(3)}(x) + a_1\Phi_1^{(3)}(x) + a_2\Phi_2^{(3)}(x) + ... + a_N\Phi_N^{(3)}(x)] \\ +P_1[a_0\Phi_0^{(2)}(x) + a_1\Phi_1^{(2)}(x) + a_2\Phi_2^{(2)}(x).... + a_N\Phi_N^{(2)}(x)] \\ +P_2[a_0\Phi_0^{(1)}(x) + a_1\Phi_1^{(1)}(x) + a_2\Phi_2^{(1)}(x) + ... + a_N\Phi_N^{(1)}(x)] \\ +P_3[a_0\Phi_0^{(0)}(x) + a_1\Phi_1^{(0)}(x) + a_2\Phi_2^{(0)}(x) + ... + a_N\Phi_N^{(0)}(x)] \\ +\mathbf{C}(X) + \mathbf{G}_2(X,t) = f(x) \end{gathered} \tag{33}$$

Here,

$$\Phi_N^{(3)}(x) = \int \int \int \Phi_N dx dx dx \tag{34}$$

$$\Phi_0^{(3)}(x) = \int \int \int \Phi_0 dx dx dx \tag{35}$$

$$\mathbf{G}_2(X,t) = \int_a^b k(x,t) y_{N,n}(t) dt \tag{36}$$

After evaluating the terms involving integrals in equation (33) and with further simplification, we then collocated the left-over at the point $x = x_k$, we obtained

$$P_0[a_0\Phi_0^{(3)}(x_k) + a_1\Phi_1^{(3)}(x_k) + a_2\Phi_2^{(3)}(x_k) + ... + a_N\Phi_N^{(3)}(x_k)]$$
$$+P_1[a_0\Phi_0^{(2)}(x_k) + a_1\Phi_1^{(2)}(x_k) + a_2\Phi_2^{(2)}(x_k) + ... + a_N\Phi_N^{(2)}(x_k)]$$
$$+P_2[a_0\Phi_0^{(1)}(x_k) + a_1\Phi_1^{(1)}(x_k) + a_2\Phi_2^{(1)}(x_k) + ... + a_N\Phi_N^{(1)}(x_k)]$$
$$+P_3[a_0\Phi_0^{(0)}(x_k) + a_1\Phi_1^{(0)}(x_k) + a_2\Phi_2^{(0)}(x_k) + ... + a_N\Phi_N^{(0)}(x_k)]$$
$$+\mathbf{C}(X_k) + \mathbf{G}_2(X_k, t) = f(x_k) \tag{37}$$

where,

$$x_k = a + \frac{(b-a)k}{N+2}, \quad k \ = \ 1,\ 2,\ 3, ..., N+1 \tag{38}$$

Putting equation (38) into (37), we obtained (N+1) algebraic equations with (N+4) unknown constants. Three extra equations are obtained using the initial/boundary conditions given in equations (3) and (4). Altogether, we have (N+4) algebraic equations with (N+4) unknown constants. This system of (N+4) algebraic linear equations is put in vector form as $A\underline{X} = \underline{b}$ and then solved using Gaussian 'Maple 18' software to obtain the unknown constants a_j $(j \geq 0)$ and $c_i's$. These values are then substituted into our assumed solution to obtain the approximate solution.

3.0 Error Analysis

In this section, we presented the error analysis and the asymptotic error estimate of the method.

Theorem: Let $f(x)$ have two continuous derivatives on the interval $a \leq x \leq b$, then

$$E_n^T(f) \equiv \int_a^b f(x)dx - T_n(f) = -\frac{h^2(b-a)f''(c_n)}{12} \tag{39}$$

for some $a \leq c_n \leq b$ [12, 13].

Following [12, 13, 14], the error in the Integrated Trapezoidal method with only a single subinterval, is given by

$$E_{N,n}^T(y_{N,n}) \equiv \int_{x_0}^{x_0+h} y_{N,n}(x)dx - \frac{h}{2}[y_{N,n}(x_0) + y_{N,n}(x_0+h)] = -\frac{h^3}{12}y''_{N,n}(c) \tag{40}$$

where, N is the degree of the approximating polynomial and $x_0 \leq c \leq (x_0 + h)$.

Recall that the general trapezoidal rule for $y_{N,n}(x)$ was obtained by dividing the entire interval into n-subintervals and then applying the simple trapezoidal rule on each subinterval, hence, we have

$$I^T(y_{N,n}) = \int_{x_0}^{x_n} y_{N,n}(x)dx$$

$$= \int_{x_0}^{x_1} y_{N,n}(x)dx + \int_{x_1}^{x_2} y_{N,n}(x)dx + ... + \int_{x_{n-1}}^{x_n} y_{N,n}(x)dx \tag{41}$$

$$I^T(y_{N,n}) \approx \frac{h}{2}[y_{N,n}(x_0) + y_{N,n}(x_1)] + \frac{h}{2}[y_{N,n}(x_1) + y_{N,n}(x_2)] + ... + \frac{h}{2}[y_{N,n}(x_{n-1}) + y_{N,n}(x_n)] \tag{42}$$

where,

$$h = \frac{x_n - x_0}{n}$$

$x_i = x_0 + ih,\ i = 0,\ 1, ..., n.$

Then, the error

$$E^T_{N,n}(y_{N,n}) \equiv \int_{x_0}^{x_n} y_{N,n}(x)dx - I^T(y_{N,n}) \tag{43}$$

can be obtained by adding together the errors over the n-subintervals.

Since

$$\int_{x_0}^{x_0+h} y_{N,n}(x)dx - \frac{h}{2}[y_{N,n}(x_0) + y_{N,n}(x_0 + h)] = -\frac{h^3}{12}y''_{N,n}(c)$$

then on a general interval, $[x_{i-1},\ x_i]$, the error is

$$\int_{x_{i-1}}^{x_i} y_{N,n}(x)dx - \frac{h}{2}[y_{N,n}(x_{i-1}) + y_{N,n}(x_i)] = -\frac{h^3}{12}y''_{N,n}(c_i)$$

with $x_{i-1} \leq c_i \leq x_i$.

Then, the errors in all the n-subintervals put together, is given by

$$E^T_{N,n}(y_{N,n}) = -\frac{h^3}{12}y''_{N,n}(c_1) - \frac{h^3}{12}y''_{N,n}(c_2) - ... - \frac{h^3}{12}y''_{N,n}(c_n) \tag{44}$$

$$= -\frac{h^3 n}{12}[\frac{y''_{N,n}(c_1) + y''_{N,n}(c_2) + ... + y''_{N,n}(c_n)}{n}]$$

$$= -\frac{h^3 n}{12}\alpha_n \tag{45}$$

where, α_n denotes $[\frac{y''_{N,n}(c_1)+y''_{N,n}(c_2)+...+y''_{N,n}(c_n)}{n}]$, such that

$$\min_{x_0 \leq x \leq x_n} y''_{N,n}(x) \leq \alpha_n \leq \max_{x_0 \leq x \leq x_n} y''_{N,n}(x)$$

Following [12, 15], we assumed that $y''_{N,n}(x)$ is a continuous function, then there is a number k_n in $[x_0,\ x_n]$ for which

$$y''_{N,n}(k_n) = \alpha_n \tag{46}$$

Therefore,

$$E^T_{N,n}(y_{N,n}) = -\frac{h^3 n\alpha_n}{12} = -\frac{h^3 n}{12}y''_{N,n}(k_n)$$

$$= -\frac{h^2(x_n - x_0)}{12}y''_{N,n}(k_n) \tag{47}$$

since

$$h = \frac{x_n - x_0}{n}$$

3.1 Asymptotic Error Estimate

Equation (44) is re-written as

$$E^T_{N,n}(y_{N,n}) = -\frac{h^2}{12}[y''_{N,n}(c_1)h + y''_{N,n}(c_2)h + ... + y''_{N,n}(c_n)h] \tag{48}$$

If

$$\lim_{n\longrightarrow\infty}[y''_{N,n}(c_1)h + y''_{N,n}(c_2)h + ... + y''_{N,n}(c_n)h] = \int_{x_0}^{x_n} y''_{N,n}(x)dx \tag{49}$$

Then,

$$\int_{x_0}^{x_n} y''_{N,n}(x)dx = y'_{N,n}(x_n) - y'_{N,n}(x_0) \tag{50}$$

where,

$$y''_{N,n}(c_1)h + y''_{N,n}(c_2)h + ... + y''_{N,n}(c_n)h$$

is a Riemann sum for the integral given by equation (50) [12, 14]. Thus,

$$y''_{N,n}(c_1)h + y''_{N,n}(c_2)h + ... + y''_{N,n}(c_n)h \approx y'_{N,n}(x_n) - y'_{N,n}(x_0) \tag{51}$$

for large values of n.

Substituting equation (48) into equation (51), the asymptotic error estimate in the method is given by

$$\begin{aligned} E^T_{N,n}(y_{N,n}) &\approx -\frac{h^2}{12}[y'_{N,n}(x_n) - y'_{N,n}(x_0)] \\ &\equiv \widehat{E}^T_{N,n}(y_{N,n}) \end{aligned} \tag{52}$$

4.0 Numerical Examples

In this section, we demonstrated the Integrated Trapezoidal Collocation Method by the two basis functions considered in this paper to solve some linear and non-linear third order Integro-differential equations. The results obtained by the two basis functions were compared with the results by Chebyshev polynomials.

Remark: We defined error used as

$$\text{Error} = |y(x) - y_{N,n}(x)| \;:\; a \le x \le b \text{ for } \; N = 1,\ 2,\ 3, ...$$

such that $n \ge 2$ is an integer.

Example 1: Consider the third order pseudo-linear Fredholm Integro-Differential Equation

$$y'''(x) = y''(x) - x + \int_0^1 Sin(x)e^{-t}y(t)dt \tag{39}$$

subject to the boundary conditions

$$y(0) = 1, \;\; y'(0) = 0, \;\; y'(1) = 1.$$

The exact solution of this problem is

$$y(x) = 0.6799 - 0.3200x - 0.5000x^2 + \frac{1}{6}x^3 - 0.0276e^x + 0.3477(Sinx + Cosx)$$

Remark: Using Canonical Polynomials as the bases functions, we compared equation (39) with equation (16) and with slight modification, we have:

$P_0 = 1$, $P_1 = 0$, $P_2 = -1$ and $P_3 = 1$

Therefore, the first-few Canonical Polynomials for this problem are:

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$$
\text{when} \quad \begin{aligned}
r &= 0, \quad \Phi_0(x) = 1 \\
r &= 1, \quad \Phi_1(x) = x \\
r &= 2, \quad \Phi_2(x) = x^2 + 2 \\
r &= 3, \quad \Phi_3(x) = x^3 + 6x - 6 \\
r &= 4, \quad \Phi_4(x) = x^4 + 12x^2 - 24x + 24
\end{aligned}
$$

Example 2: Consider the third order linear Fredholm Integro-Differential Equation

$$
y'''(x) = Sin(x) + x + \int_0^{\frac{\pi}{2}} xty'(t)dt, \quad 0 \le x \le 1 \tag{40}
$$

subject to the initial conditions

$$
y(0) = 1, \;\; y'(0) = 0, \;\; y''(0) = -1.
$$

The exact solution of this problem is

$$
y(x) = Cos(x).
$$

Example 3: Consider the third order non-linear Fredholm-Volterra Integro-Differential Equation

$$
y'''(x) = f(x) - y(x) + \int_{-1}^{1} (x^2 + xt^2) y^2(t)dt + \int_{-1}^{x} y^2(t)dt, \qquad 0 \le x \le 1 \tag{41}
$$

where,

$$
f(x) = \frac{47}{14} - \frac{17}{9}x + \frac{4}{5}x^2 + x^3 + \frac{1}{2}x^4 - \frac{1}{7}x^7,
$$

subject to the initial conditions

$$
y(0) = -1, \quad y'(0) = 0 \text{ and } y''(0) = 0.
$$

The exact solution of this problem is

$$
y(x) = x^3 - 1.
$$

Tables of Results

Table 1: Numerical Results for Example 1

x	Exact	Integrated Trapezoidal Collocation Method: $n = 8$					
		Power Series		Canonical Polynomials		Chebyshev Polynomials	
		N=4	Error	N=4	Error	N=4	Error
0.0	1.00000	1.00000	0.00000	1.00000	0.00000	1.00000	0.00000
0.1	1.00324	1.00335	1.10E-4	1.00335	1.10E-4	1.00335	1.10E-4
0.2	1.01337	1.01385	4.80E-4	1.01385	4.80E-4	1.01385	4.80E-4
0.3	1.03107	1.03217	1.10E-3	1.03217	1.10E-3	1.03217	1.10E-3
0.4	1.05705	1.05901	1.96E-3	1.05901	1.96E-3	1.05901	1.96E-3
0.5	1.09206	1.09508	3.02E-3	1.09508	3.02E-3	1.09508	3.02E-3
0.6	1.13690	1.14111	4.21E-3	1.14111	4.21E-3	1.14111	4.21E-3
0.7	1.19242	1.19785	5.43E-3	1.19785	5.43E-3	1.19785	5.43E-3
0.8	1.25948	1.26606	6.58E-3	1.26606	6.58E-3	1.26606	6.58E-3
0.9	1.33901	1.34652	7.51E-3	1.34652	7.51E-3	1.34652	7.51E-3
1.0	1.43198	1.44003	8.05E-3	1.44003	8.05E-3	1.44003	8.05E-3

Table 2: Numerical Results for Example 2

x	Exact	Integrated Trapezoidal Collocation Method: $n = 8$					
		Power Series		Canonical Polynomials		Chebyshev Polynomials	
		N=4	Error	N=4	Error	N=4	Error
0.0	1.000000	1.000000	0.000000	1.000000	0.000000	1.000000	0.000000
0.1	0.999999	0.999946	5.3002E-5	0.999946	5.3000E-5	0.999958	4.100E-5
0.2	0.999995	0.999937	5.8000E-5	0.999937	5.8000E-5	0.999948	4.700E-5
0.3	0.999989	0.999917	7.2010E-5	0.999917	7.2000E-5	0.999936	5.300E-5
0.4	0.999980	0.999904	7.6000E-5	0.999904	7.6000E-5	0.999916	6.400E-5
0.5	0.999969	0.999882	8.7000E-5	0.999882	8.7000E-5	0.999887	8.200E-5
0.6	0.999956	0.999867	8.9020E-5	0.999867	8.9000E-5	0.999871	8.500E-5
0.7	0.999940	0.999847	9.3300E-5	0.999847	9.3300E-5	0.999853	8.700E-5
0.8	0.999921	0.999822	9.9000E-5	0.999822	9.9000E-5	0.999826	9.500E-5
0.9	0.999900	0.999480	4.2000E-4	0.999480	4.2000E-4	0.999280	6.200E-4
1.0	0.999877	0.999317	5.6000E-4	0.999317	5.6000E-4	0.999237	6.400E-4

Table 3: Numerical Results for Example 3

x	Exact	Integrated Trapezoidal Collocation Method: $n = 8$					
		Power Series		Canonical Polynomials		Chebyshev Polynomials	
		N=4	Error	N=4	Error	N=4	Error
0.0	-1.00000	-1.00000	0.00000	-1.00000	0.00000	-1.00000	0.000000
0.1	-0.99900	-0.99534	3.66E-3	-0.99686	2.14E-3	-0.99685	2.15E-3
0.2	-0.99200	-0.98716	4.84E-3	-0.98808	3.92E-3	-0.98806	3.94E-3
0.3	-0.97300	-0.96240	2.19E-3	-0.97097	2.03E-3	-0.97093	2.07E-3
0.4	-0.93600	-0.93792	1.92E-3	-0.93476	1.24E-3	-0.93472	1.28E-3
0.5	-0.87500	-0.85310	8.02E-3	-0.86884	6.16E-3	-0.86879	6.21E-3
0.6	-0.78400	-0.78310	9.00E-4	-0.78317	8.30E-4	-0.78317	8.30E-4
0.7	-0.65700	-0.63200	2.50E-4	-0.65681	1.90E-4	-0.65681	1.90E-4
0.8	-0.48800	-0.48816	1.60E-4	-0.48788	1.20E-4	-0.48788	1.20E-4
0.9	-0.27100	-0.26702	3.98E-3	-0.26869	2.31E-3	-0.26869	2.31E-3
1.0	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000

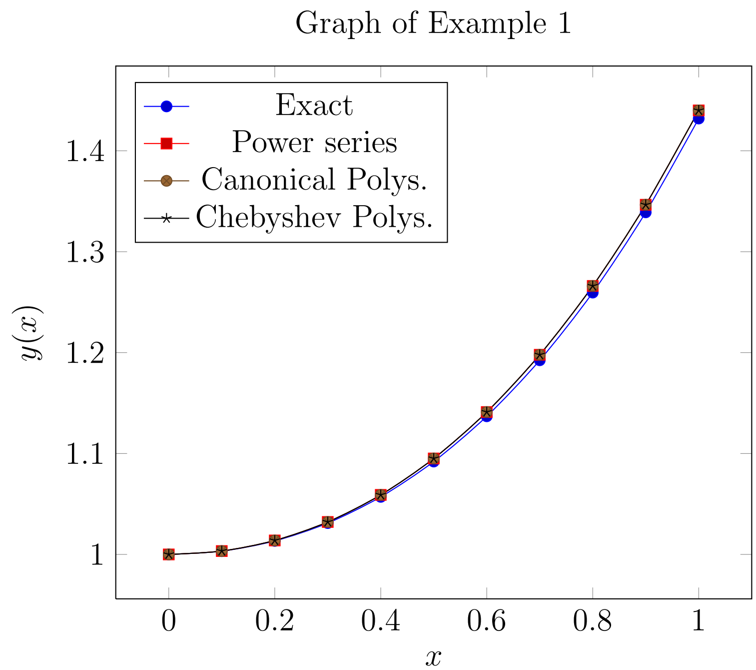

Figure 1: The behaviour of the exact solution and the approximate solutions using Power series, Canonical polynomials and Chebyshev polynomials as basis functions at $N = 4$ and $n = 8$.

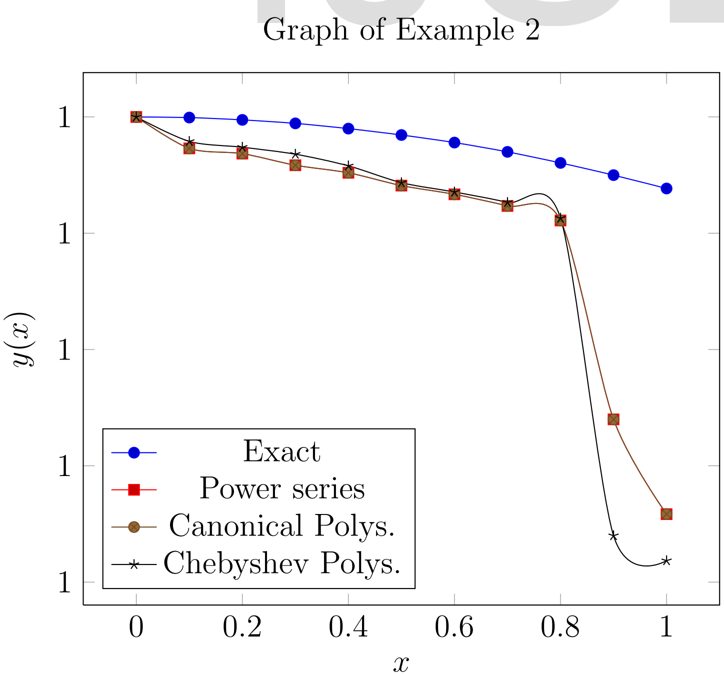

Figure 2: The behaviour of the exact solution and the approximate solutions using Power series, Canonical polynomials and Chebyshev polynomials as basis functions at $N = 4$ and $n = 8$.

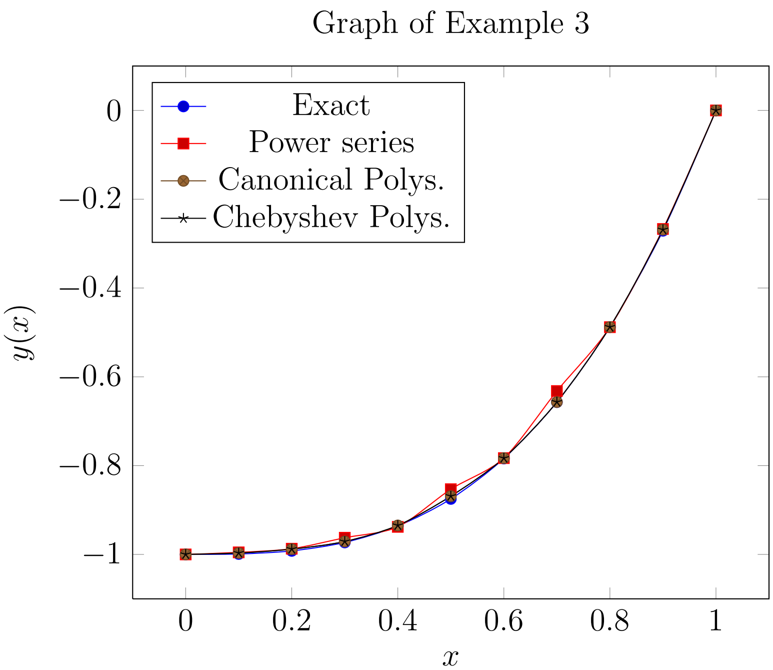

Figure 3: The behaviour of the exact solution and the approximate solutions using Power series, Canonical polynomials and Chebyshev polynomials as basis functions at $N = 4$ and $n = 8$.

4.0 Discussion of Results and Conclusion

In this paper, we have shown that both the Power Series and Canonical Polynomial Integrated Trapezoidal Collocation method can efficiently solve linear and non-linear third-order Integro-Differential Equations with high accuracy. Moreover, the results obtained by Power series are in close agreement with the results obtained by Canonical and Chebyshev Polynomials. In conclusion, the method yields the desired accuracy when the results are compared with the exact solutions as shown in Tables 1-3 and Figures 1-3.

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Authors' Declaration:

The author hereby declares that there is no conflict of interest regarding the publication of this paper.